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Generalized Phase Space Method in Spin Systems—Spin Coherent State Representation

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A generalized phase space method for spin operators is developed. With the use of a spin coherent state representation, mapping rules from spin operators onto a c-number space are established; simple formulas to calculate the mapped c-number functions are also derived. A product theorem, which gives a way of mapping a product of operators, is obtained in an intuitive form. This can be advantageously used to transform a Liouville equation into a c-number equation. As an illustrative example, the method is applied to the Heisenberg model of a magnet.

KEY WORDS: Generalized phase space; mapping rules; product theorem; quasi-probability function.

1. INTRODUCTION

There have been several attempts to describe quantum mechanical systems in terms of c-number functions.[1,2] In the case of a Bose field, phase space methods,[2-4] where Bose operators are mapped onto a c-number space, have played an important role. Making use of a quasi-probability distribution

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function mapped from a density matrix, we can construct a theoretical framework which is quite analogous to that of probability theory. Quantum mechanical and stochastic averages can thus be treated on an equal footing. For Bose systems, some of these mapping methods are closely related to the coherent state representation.[5]

It has been recognized that similar treatments[6,7] are possible also for spin systems. However, the existent methods are not so transparent as in the case of boson systems. Our aim here is to present a systematic analysis of the problem which, we hope, sheds a new light on the problem and paves the way for wider applications. For this purpose, we use a method of spin coherent state representation developed in a previous paper.[8]

In Section 2, we consider basic properties of the spin coherent state. The mapping rules of operators onto a c-number space are also described. In Section 3 we treat transformation properties of the spin coherent state and its relation to the Bloch state.[9,10] A product theorem, by which a product of two operators is mapped onto a c-number space, is established in Section 4. By using the theorem, we give a c-number form of the Liouville equation (and the Heisenberg equation of motion) in Section 5, where the classical limit is also examined. In the same section, a simple c-number description of the Heisenberg magnet is obtained. In Section 6, we present some discussions.

2. ORDERED OPERATOR EXPANSION FOR SPIN OPERATORS

The method of mapping quantum mechanical operators onto a c-number space is closely related to the ordering of these noncommuting operators.[3,4] For the Bose operators a and $a^\dagger$, we have three kinds of ordering by expressing an arbitrary operator G in the form

$$G = \sum_{m,n} g_{mn}^{(N)}(a^\dagger)^m a^n \qquad \text{(normal order)} \tag{1}$$

$$G = \sum_{m,n} g_{mn}^{(A)} a^n (a^\dagger)^m \qquad \text{(anti-normal order)} \tag{2}$$

and

$$G = \sum_{m,n} g_{mn}^{(W)}\{(a^\dagger)^m a^n\}_W \quad \text{(symmetrized or Weyl order)} \tag{3}$$

Then we have a corresponding c-number function $F^{(\Omega)}(z, z^*)$ (Ω = N, A, W) by replacing the ordered operators $(a^\dagger)^m a^n$, $a^n(a^\dagger)^m$, and $\{(a^\dagger)^m a^n\}_W$ by a monomial $(z^*)^m z^n$.

In the following, we shall investigate a method of mapping spin operators onto a c-number space.

According to Schwinger,[11] spin operators $\{S_+, S_-, \text{ and } S_z\}$ are expressed in terms of Bose operators $\{a_+, a_+^\dagger\}$ and $\{a_-, a_-^\dagger\}$ as

$$S_+ = a_+^\dagger a_- \tag{4a}$$

$$S_- = a_-^\dagger a_+ \tag{4b}$$

and

$$S_z = \tfrac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) \tag{4c}$$

With the use of these relations, any operator G can be regarded as a function of operators $\{a_+, a_+^\dagger\}$ and $\{a_-, a_-^\dagger\}$.

A spin coherent state is defined as the eigenstate of the two kinds of Bose annihilation operators a_+ and a_-:

$$a_+|\mathbf{z}\rangle = z_+|\mathbf{z}\rangle \tag{5a}$$

$$a_-|\mathbf{z}\rangle = z_-|\mathbf{z}\rangle \tag{5b}$$

where $\mathbf{z}$ denotes a set of two complex numbers $\{z_+, z_-\}$. The state $|\mathbf{z}\rangle$ can be expanded in terms of the angular momentum eigenstates $|J, m)$ as[2]

$$|\mathbf{z}\rangle = \sum_J \sum_{m=-J}^{J} \left(\exp -\frac{|z|^2}{2}\right) \frac{z_+^{J+m} z_-^{J-m}}{[(J+m)!\,(J-m)!]^{1/2}} \left|J, m\right) \tag{6}$$

where

$$|z|^2 = |z_+|^2 + |z_-|^2$$

Our task is now to construct the mapping rules for spin operators. In other words, we must determine the expansion coefficients similar to $g_{mn}^{(\Omega)}$ appearing in (1)–(3). This is achieved by introducing a displacement operator $D(\mathbf{z})$ defined by

$$D(z) = \exp(\mathbf{z}\mathbf{a}^\dagger - \mathbf{z}^*\mathbf{a}) \tag{7}$$

which generates the state $|\mathbf{z}\rangle$ as

$$|\mathbf{z}\rangle = D(\mathbf{z})|0\rangle \tag{8}$$

Because of the completeness of the displacement operator $D(\boldsymbol{\alpha})$, an operator G can be expanded in the form

$$G = \int \frac{d^2\boldsymbol{\alpha}}{\pi^2}\, g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) D(\boldsymbol{\alpha}) \tag{9}$$

where $g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$ is given by

$$g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \operatorname{Tr} G D(-\boldsymbol{\alpha}) \tag{10}$$

[2] The angular momentum eigenstate $|J, m)$ is defined as the eigenstate of number operators $a_+^\dagger a_+$ and $a_-^\dagger a_-$ with eigenvalues $J + m$ and $J - m$, respectively, i.e., $|J + m, J - m\rangle$.

The displacement operator $D(\boldsymbol{\alpha})$ can be expanded in various forms:

$$D(\boldsymbol{\alpha}) = \left(\exp - \frac{|\alpha|^2}{2}\right) \sum_{\substack{n_1 n_2 \\ m_1 m_2}} \frac{(\alpha_+)^{n_1}(-\alpha_+{}^*)^{n_2}(\alpha_-)^{m_1}(-\alpha_-{}^*)^{m_2}}{n_1!\, n_2!\, m_1!\, m_2!} \times (a_+{}^\dagger)^{n_1}(a_+)^{n_2}(a_-{}^\dagger)^{m_1}(a_-)^{m_2} \quad \text{(normal)} \tag{11a}$$

$$D(\boldsymbol{\alpha}) = \left(\exp \frac{|\alpha|^2}{2}\right) \sum_{\substack{n_1 n_2 \\ m_1 m_2}} \frac{(\alpha_+)^{n_1}(-\alpha_+{}^*)^{n_2}(\alpha_-)^{m_1}(-\alpha_-{}^*)^{m_2}}{n_1!\, n_2!\, m_1!\, m_2!} \times (a_+)^{n_2}(a_+{}^\dagger)^{n_1}(a_-)^{m_2}(a_-{}^\dagger)^{m_1} \quad \text{(anti-normal)} \tag{11b}$$

$$D(\boldsymbol{\alpha}) = \sum_{\substack{n_1 n_2 \\ m_1 m_2}} \frac{(\alpha_+)^{n_1}(-\alpha_+{}^*)^{n_2}(\alpha_-)^{m_1}(-\alpha_-{}^*)^{m_2}}{n_1!\, n_2!\, m_1!\, m_2!} \times (a_+^{\dagger n_1} a_+^{n_2} a_-^{m_1} a_-^{\dagger m_2})_W \quad \text{(symmetrized)} \tag{11c}$$

where $(a_+^{\dagger n_1} a_+^{n_2} a_-^{\dagger m_1} a_-^{m_2})_W$ is given by

$$\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial(\alpha_+)^{n_1}\,\partial(\alpha_+{}^*)^{n_2}\,\partial(\alpha_-)^{m_1}\,\partial(\alpha_-{}^*)^{m_2}} D(\boldsymbol{\alpha})\bigg|_{\alpha=0}$$

The operator G is, therefore, expanded as

$$G = \sum g^{(\mathrm{N})}_{n_1 n_2 m_1 m_2} a_+^{\dagger n_1} a_+^{n_2} a_-^{\dagger m_1} a_-^{m_2} \tag{12a}$$

in the normal order, and

$$G = \sum g^{(\mathrm{A})}_{n_1 n_2 m_1 m_2} a_+^{n_2} a_+^{\dagger n_1} a_-^{m_2} a_-^{\dagger\ m_1} \tag{12b}$$

in the anti-normal order, where the coefficients g are given by

$$g^{(\mathrm{N})}_{n_1 n_2 m_1 m_2} = \int \frac{d^2\boldsymbol{\alpha}}{\pi^2} g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)\left(\exp - \frac{|\alpha|^2}{2}\right) \frac{(\alpha_+)^{n_1}(-\alpha_+{}^*)^{n_2}(\alpha_-)^{m_1}(-\alpha_-{}^*)^{m_2}}{n_1!\, n_2!\, m_1!\, m_2!} \tag{13a}$$

and

$$g^{(\mathrm{A})}_{n_1 n_2 m_1 m_2} = \int \frac{d^2\boldsymbol{\alpha}}{\pi^2} g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)\left(\exp \frac{|\alpha|^2}{2}\right) \frac{(\alpha_+)^{n_1}(-\alpha_+{}^*)^{n_2}(\alpha_-)^{m_1}(-\alpha_-{}^*)^{m_2}}{n_1!\, n_2!\, m_1!\, m_2!} \tag{13b}$$

Now we consider a complex function $F(\mathbf{z}, \mathbf{z}^*)$, which can be written in the form

$$\begin{aligned} F(\mathbf{z}, \mathbf{z}^*) &= \int \frac{d^2\boldsymbol{\alpha}}{\pi^2} f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) \exp(\boldsymbol{\alpha}\mathbf{z}^* - \boldsymbol{\alpha}^*\mathbf{z}) \\ &= \sum \int \frac{d^2\boldsymbol{\alpha}}{\pi^2} f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) \frac{(\alpha_+)^{n_1}(-\alpha_+{}^*)^{n_2}(\alpha_-)^{m_1}(-\alpha_-{}^*)^{m_2}}{n_1!\, n_2!\, m_1!\, m_2!} \\ &\quad \times z_+^{*n_1} z_+^{n_2} z_-^{*m_1} z_-^{m_2} \end{aligned} \tag{14}$$

By comparing (13) and (14), we can map the operator G onto the c-number function $F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$, if the spectral functions $g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$ and $f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$ are related each other by

$$g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = (\exp \tfrac{1}{2}|\alpha|^2) f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) \tag{15a}$$

for the normal ordering, and by

$$g(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = (\exp - \tfrac{1}{2}|\alpha|^2) f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) \tag{15b}$$

for the anti-normal ordering.

In the boson case, the relations (15a) and (15b) have been generalized to the parametrized form

$$g(\alpha, \alpha^*) = [\exp(\tfrac{1}{2}s|\alpha|^2)] f(\alpha, \alpha^*)$$

by Cahill and Glauber,[3] while Agarwal and Wolf[4] have discussed the mapping in a somewhat different form,

$$g(\alpha, \alpha^*) = \Omega(\alpha, \alpha^*) f(\alpha, \alpha^*)$$

where $\Omega(\alpha, \alpha^*)$ is called the filter function.

To summarize, the operator G and the c-number function $F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ are related to each other through the formulas

$$G = \int \frac{d^2\mathbf{z}}{\pi^2} F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*) \Delta^{(\Omega)}(\mathbf{z} - \boldsymbol{\alpha}, \mathbf{z}^* - \boldsymbol{\alpha}^\dagger) \tag{16}$$

and

$$F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*) = \operatorname{Tr} G\, \Delta^{(\tilde{\Omega})}(\mathbf{z} - \boldsymbol{\alpha}, \mathbf{z}^* - \boldsymbol{\alpha}^\dagger) \tag{17}$$

where $\Delta^{(\Omega)}(\mathbf{z} - \mathbf{a}, \mathbf{z}^* - \mathbf{a}^\dagger)$, the mapping delta operator, is given by

$$\Delta^{(\Omega)}(\mathbf{z} - \boldsymbol{\alpha}, \mathbf{z}^* - \boldsymbol{\alpha}^\dagger) = \int \frac{d^2\boldsymbol{\alpha}}{\pi^2} \Omega(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) [\exp(\mathbf{z}\boldsymbol{\alpha}^* - \mathbf{z}^*\boldsymbol{\alpha})] D(\boldsymbol{\alpha}) \tag{18}$$

The superscript on $\tilde{\Omega}$ specifies the anti-reciprocal mapping to Ω, its filter function being given by

$$\tilde{\Omega}(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \Omega^{-1}(-\boldsymbol{\alpha}, -\boldsymbol{\alpha}^*) \tag{19}$$

For the normal and anti-normal ordering of operators, the functions $F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ are given by

$$F^{(N)}(\mathbf{z}, \mathbf{z}^*) = \langle \mathbf{z}|G|\mathbf{z}\rangle \tag{20a}$$

and

$$F^{(A)}(\mathbf{z}, \mathbf{z}^*) = \frac{1}{\pi^2} [\exp(|z|^2)] \int \frac{d^2\boldsymbol{\alpha}}{\pi^2} \langle -\boldsymbol{\alpha}|G|\boldsymbol{\alpha}\rangle \exp(|\alpha|^2) \exp(\boldsymbol{\alpha}^*\mathbf{z} - \boldsymbol{\alpha}\mathbf{z}^*) \tag{20b}$$

respectively.

Here we must take into account the fact that the Bose operators $\{a_\pm, a_\pm{}^\dagger\}$ appear in our problems through the form of (4). Then, fixing the total number of bosons is equivalent to specifying a particular irreducible representation of the rotation group. It is, therefore, important to consider the transformation properties of the spin coherent state under rotations.

3. TRANSFORMATION PROPERTIES OF THE SPIN COHERENT STATE UNDER ROTATION

In order to relate the description in terms of the complex numbers z_+ and z_- with that in the polar and azimuthal angles θ and ϕ, it is essential to know the transformation properties of the spin coherent state.

Any rotation of coordinate system is characterized by the successive Euler rotations (α, β, γ)—a rotation made about the z axis through an angle α, followed by a rotation about the new, transformed y' axis through an angle β, and finally a rotation about the z'' axis, (z axis rotated around y' axis) through an angle γ. After these rotations, a state $|\,\rangle$ is transformed into a new state $|\,\rangle'$:

$$|\,\rangle' = R(\alpha, \beta, \gamma)|\,\rangle \tag{21}$$

where

$$R(\alpha, \beta, \gamma) = e^{i\gamma S_z} e^{i\beta S_y} e^{i\alpha S_z} \tag{22}$$

Since a set of operators $\{a_+{}^\dagger, a_-{}^\dagger\}$ forms an irreducible tensor of rank $\frac{1}{2}$ with respect to spin operators defined in (4), they transform under rotation as

$$\begin{aligned} e^{i\theta S_z} a_+{}^\dagger e^{-i\theta S_z} &= e^{i\theta/2} a_+{}^\dagger \\ e^{i\theta S_z} a_-{}^\dagger e^{-i\theta S_z} &= e^{-i\theta/2} a_-{}^\dagger \\ e^{i\theta S_y} a_+{}^\dagger e^{-i\theta S_y} &= (\cos \tfrac{1}{2}\theta) a_+{}^\dagger - (\sin \tfrac{1}{2}\theta) a_-{}^\dagger \end{aligned} \tag{23}$$

and

$$e^{i\theta S_y} a_-{}^\dagger e^{-i\theta S_y} = (\sin \tfrac{1}{2}\theta) a_+{}^\dagger + (\cos \tfrac{1}{2}\theta) a_-{}^\dagger$$

Due to these transformation properties, the complex numbers $\mathbf{z}$ transform as covariant components of a spinor. Thus we have

$$R(\alpha, \beta, \gamma)|\mathbf{z}\rangle = |\mathscr{R}\mathbf{z}\rangle \tag{24}$$

where $\mathscr{R}$ is a two by two matrix given by

$$\mathscr{R} = e^{i\gamma\sigma_z/2} e^{i\beta\sigma_y/2} e^{i\alpha\sigma_z/2}$$

the σ_μ being the Pauli matrices.

In order to relate the complex numbers z_+ and z_- with the polar and azimuthal angles θ and ϕ, we first consider a spin coherent state $|z, 0\rangle$

where z is a complex number. The state $|z, 0\rangle$ can be expanded in terms of the angular momentum eigenstates with the maximum eigenvalue of S_z:

$$|z, 0\rangle = \sum_{J=0}^{\infty} (\exp -\tfrac{1}{2}|z|^2)\{z^{2J}/[(2J)!]^{1/2}\}|J, J) \tag{25}$$

By acting with the rotation operator $R(-\psi, -\theta, -\phi)$ on this state, we find a transformed quantization axis of spin specified by the polar and azimuthal angles θ and ϕ. The transformed state is also a spin coherent state and is represented by

$$|z_+, z_-\rangle = R(-\psi, -\theta, -\phi)|z, 0\rangle$$

where

$$z_+ = ze^{-i\psi/2}e^{-i\phi/2}\cos(\theta/2), \qquad z_- = ze^{-i\psi/2}e^{i\phi/2}\sin(\theta/2)$$

The common phase factor $e^{-i\psi/2}$ can be absorbed ino the phase of the complex number z, and therefore it is sufficient to only consider the operator $R(0, -\theta, -\phi)$, which will be denoted $R(\omega)$ or $R(\theta, \phi)$ hereafter.

Thus any spin coherent state $|\mathbf{z}\rangle$ is generated by acting with a suitable transformation operator $R(\theta, \phi)$ on a state $|z, 0\rangle$:

$$|z_+, z_- > = R(\theta, \phi)|z, 0\rangle \tag{26}$$

where complex numbers z_+ and z_- are given by

$$z_+ = ze^{-i\phi/2}\cos(\theta/2) \tag{27a}$$

and

$$z_- = ze^{i\phi/2}\sin(\theta/2) \tag{27b}$$

Because $R(\theta, \phi)$ commutes with the total number operator n of bosons, (26) holds for each subspace of a fixed magnitude of the spin. Then we have

$$|\mathbf{z}\rangle = \sum_J |J; \mathbf{z}\rangle$$

$$\begin{aligned} |J; \mathbf{z}\rangle &= R(\omega)|J; z, 0\rangle \\ &= (\exp -\tfrac{1}{2}|z|^2)\{z^{2J}/[(2J)!]^{1/2}\}R(\omega)|J, J) \\ &= (\exp -\tfrac{1}{2}|z|^2)\{z^{2J}/[(2J)!]^{1/2}\}|J; \omega\rangle \end{aligned} \tag{28}$$

and

$$|J; z, 0\rangle = (\exp -|z|^2)\{z^{2J}/[(2J)!]^{1/2}\}|J, J) \tag{29}$$

The state $|J; \omega\rangle$ is called the Bloch state:

$$|J; \omega\rangle = R(\theta, \phi)|J, J) \tag{30}$$

We must note that the subspace with a fixed magnitude of spin J is the component proportional to $(\exp -\tfrac{1}{2}|z|^2)\{z^{2J}/[(2J)!]^{1/2}\}$ in the spin coherent state

$|\mathbf{z}\rangle$. Completeness of the spin coherent state is represented by

$$\int \frac{d^2\mathbf{z}}{\pi^2} |\mathbf{z}\rangle\langle\mathbf{z}| = \int \frac{|z|^2\, d^2z\, d\omega}{8\pi} |\mathbf{z}\rangle\langle\mathbf{z}| = 1$$

where we have used

$$\frac{d^2z_+\, d^2z_-}{\pi^2} = \frac{|z|^2}{8\pi}\, d^2z\, d\omega$$

with

$$d\omega = \sin\theta\, d\theta\, d\phi$$

By fixing the magnitude of a spin, factoring $|\mathbf{z}\rangle$ in the form of (29), and integrating over z, we obtain the completeness relation of the Bloch state[9,10]

$$(2J+1)\int \frac{d\omega}{4\pi} |J;\omega\rangle\langle J;\omega| = 1 \tag{31}$$

where we have used the fact that $|J;\omega\rangle\langle J;\omega|$ has a period of 2π instead of 4π with respect to ϕ; we designate the unit matrix in this subspace on the right-hand side by 1. Using (27), c-number equivalents of operators are expressed in terms of θ and ϕ. For the normal association rule, it is convenient to define $F_J^{(N)}(\theta,\phi)$ from the term proportional to $(\exp - |z|^2)|z|^{4J}/(2J)!$ in $F^{(N)}(\mathbf{z},\mathbf{z}^*)$ as

$$F^{(N)}(\mathbf{z},\mathbf{z}^*) = \sum_J [(\exp - |z|^2)|z|^{4J}/(2J)!]F_J^{(N)}(\theta,\phi) \tag{32}$$

which can also be expressed in the form

$$F_J^{(N)}(\theta,\phi) = \langle J;\omega|G|J;\omega\rangle \tag{33}$$

where we have used (29). For the anti-normal association rule, we define $F_J^{(A)}(\theta,\phi)$ by

$$F_J^{(A)}(\theta,\phi) = \int \frac{d^2z}{(2J+1)!} (\exp - |z|^2)|z|^{4J+2} F_J^{(A)}(\mathbf{z},\mathbf{z}^*) \tag{34}$$

Then we have the following theorem.

Theorem 1. The trace of two operators G_1 and G_2 is given by

$$\mathrm{Tr}\, G_1G_2 = \frac{2J+1}{4\pi}\int d\omega\, F_{1,J}^{(N)}(\theta,\phi)F_{2,J}^{(A)}(\theta,\phi) \tag{35}$$

(See Appendix A for the proof.)

As the operator G can be expanded in the form

$$G = \int \frac{d^2\mathbf{z}}{\pi^2} F_J^{(A)}(\mathbf{z},\mathbf{z}^*)|\mathbf{z}\rangle\langle\mathbf{z}|$$

we obtain after integrating over z

$$G = \frac{2J+1}{4\pi} \int F_J^{(A)}(\theta, \phi) |J; \omega\rangle\langle J; \omega| \, d\omega \tag{36}$$

It is seen that the function $F_J^{(A)}(\theta, \phi)$ coincides with the diagonal representation, first introduced by Arecchi *et al.*[10] Here we have obtained a simple formula for this representation. Examples of the functions $F_J^{(N)}(\theta, \phi)$ and $F_J^{(A)}(\theta, \phi)$ are calculated in Appendix B.

4. PRODUCT THEOREM

In order to obtain the phase space form of the quantum mechanical equation of motion, we have to know how the product of two operators is mapped onto a c-number space. For this purpose it is sufficient to consider the case where one of the operators is a spin operator S_μ, because we can obtain mapping rules for the product of two arbitrary operators using the result of the former case.

As a simple generalization of the Bose case,[4] we obtain the following product theorem.

Theorem 2. Let $F_1^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ and $F_2^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ be Ω-equivalents of operators G_1 and G_2; then the Ω-equivalent $F_{12}^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ of the product G_1G_2 is given by

$$\begin{aligned} F_{12}^{(\Omega)}(\mathbf{z}, \mathbf{z}^*) = {} & \exp\left\{ -\frac{1}{2}\left(\frac{\partial^2}{\partial \mathbf{z}_1{}^* \partial \mathbf{z}_2} - \frac{\partial^2}{\partial \mathbf{z}_1 \partial \mathbf{z}^*{}_2} \right) \right\} \Omega\left(\frac{\partial}{\partial \mathbf{z}_1{}^*}, -\frac{\partial}{\partial \mathbf{z}_1} \right) \\ & \times \Omega\left(\frac{\partial}{\partial \mathbf{z}_2{}^*}, -\frac{\partial}{\partial \mathbf{z}_2} \right) \tilde{\Omega}\left(\frac{\partial}{\partial \mathbf{z}_1{}^*} + \frac{\partial}{\partial \mathbf{z}_2{}^*}, -\frac{\partial}{\partial \mathbf{z}_1} - \frac{\partial}{\partial \mathbf{z}_2} \right) \\ & \times F_1^{(\Omega)}(\mathbf{z}_1, \mathbf{z}_1{}^*) F_2^{(\Omega)}(\mathbf{z}_1, \mathbf{z}_2{}^*)|_{\mathbf{z}_1 = \mathbf{z}_2 = \mathbf{z}, \mathbf{z}_1{}^* = \mathbf{z}_2{}^* = \mathbf{z}} \end{aligned} \tag{37}$$

For the cases of normal and anti-normal orderings, this is reduced to:

Corollary 1. Let $F^{(N)}(\mathbf{z}, \mathbf{z}^*)$ and $F^{(A)}(\mathbf{z}, \mathbf{z}^*)$ be c-number equivalents of an operator G, and let $\bar{F}^{(N)}(\mathbf{z}, \mathbf{z}^*)$ and $\bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*)$ be defined by

$$\begin{aligned} F^{(N)}(\mathbf{z}, \mathbf{z}^*) &= (\exp -|z|^2) \bar{F}^{(N)}(\mathbf{z}, \mathbf{z}^*) \\ F^{(A)}(\mathbf{z}, \mathbf{z}^*) &= (\exp |z|^2) \bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*) \end{aligned} \tag{38}$$

Then c-number equivalents $\bar{F}_{12}^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ of a product G_1G_2 are given by

$$\bar{F}_{12}^{(N)}(\mathbf{z}, \mathbf{z}^*) = \bar{F}_1^{(N)}\left(\frac{\partial}{\partial \mathbf{z}^*}, \mathbf{z}'^* \right) \bar{F}_2^{(N)}(\mathbf{z}, \mathbf{z}^*)|_{\mathbf{z}'^* = \mathbf{z}^*} = \bar{F}_2^{(N)}\left(\mathbf{z}', \frac{\partial}{\partial \mathbf{z}} \right) \bar{F}_1^{(N)}(\mathbf{z}, \mathbf{z}^*)|_{\mathbf{z}' = \mathbf{z}} \tag{39a}$$

and

$$\bar{F}_{12}^{(A)}(\mathbf{z}, \mathbf{z}^*) = F_1^{(A)}\left(\mathbf{z}', -\frac{\partial}{\partial \mathbf{z}}\right)\bar{F}_2^{(A)}(\mathbf{z}, \mathbf{z}^*)|_{\mathbf{z}'=\mathbf{z}} = F_2\left(-\frac{\partial}{\partial \mathbf{z}'}, \mathbf{z}'^*\right)\bar{F}_1^{(A)}(\mathbf{z}, \mathbf{z}^*)|_{\mathbf{z}'^*=\mathbf{z}^*} \tag{39b}$$

In particular, in the case where one of the operators is a spin operator S_μ, we have

$$\begin{aligned}
\bar{F}_{12}^{(N)} &= \frac{1}{2}\left(\mathbf{z}_+{}^* \frac{\partial}{\partial z_-{}^*} + \mathbf{z}_-{}^* \frac{\partial}{\partial z_+{}^*}\right)\bar{F}_2^{(N)} && \text{for} \quad G_1 = S_x \\
\bar{F}_{12}^{(N)} &= \frac{1}{2i}\left(z_+{}^* \frac{\partial}{\partial z_-{}^*} - z_-{}^* \frac{\partial}{\partial z_+{}^*}\right)\bar{F}_2^{(N)} && \text{for} \quad G_1 = S_y \\
\bar{F}_{12}^{(N)} &= \frac{1}{2}\left(z_+{}^* \frac{\partial}{\partial z_+{}^*} - z_-{}^* \frac{\partial}{\partial z_-{}^*}\right)\bar{F}_2^{(N)} && \text{for} \quad G_1 = S_z \\
\bar{F}_{12}^{(A)} &= -\frac{1}{2}\left(z_- \frac{\partial}{\partial z_+} + z_+ \frac{\partial}{\partial z_-}\right)\bar{F}_2^{(A)} && \text{for} \quad G_1 = S_x \\
\bar{F}_{12}^{(A)} &= -\frac{1}{2i}\left(z_- \frac{\partial}{\partial z_+} - z_+ \frac{\partial}{\partial z_-}\right)\bar{F}_2^{(A)} && \text{for} \quad G_1 = S_y \\
\bar{F}_{12}^{(A)} &= -\frac{1}{2}\left(z_+ \frac{\partial}{\partial z_+} - z_- \frac{\partial}{\partial z_-}\right)\bar{F}_2^{(A)} && \text{for} \quad G_1 = S_z
\end{aligned} \tag{40}$$

Mapping rules for the products GS_μ ($\mu = x, y, z$) are obtained by complex conjugation of the above expressions. A commutator $[n, G]$ is mapped onto a c-number function,

$$\frac{1}{2}\left(z^* \frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right)F^{(N)}(\mathbf{z}, \mathbf{z}^*)$$

or

$$-\frac{1}{2}\left(z^* \frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right)F^{(A)}(\mathbf{z}, \mathbf{z}^*)$$

in the phase space. Thus the condition $[n, G] = 0$ is equivalent to

$$\left(z^* \frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right)F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*) = 0 \tag{41}$$

That is, $F^{(\Omega)}(\mathbf{z}, \mathbf{z}^*)$ depends on $|z|^2$, θ, and ϕ and it is independent of the phase of the complex variable z.

Derivatives appearing in (40) can be rewritten using the *orbital angular momentum* operators defined by

$$L_+ = e^{i\phi}\left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \theta}\right), \qquad L_- = e^{-i\phi}\left(-\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi}\right) \tag{42}$$

and

$$L_z = -i\,\partial/\partial\phi$$

The results are given by[13]

$$\begin{aligned}
\frac{1}{2}\left(z_- \frac{\partial}{\partial z_+} + z_+ \frac{i}{\partial z_-}\right) &= -\frac{1}{2}L_x - \frac{i}{2}(m_y L_z - m_z L_y) + \frac{1}{2}\lambda(z, z^*)m_x \\
&\quad + \frac{1}{8}\left(e^{i\phi}\tan\frac{\theta}{2} + e^{-i\phi}\cot\frac{\theta}{2}\right)\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right) \\
\frac{1}{2i}\left(z_- \frac{\partial}{\partial z_+} - z_+ \frac{\partial}{\partial z_-}\right) &= -\frac{1}{2}L_y - \frac{i}{2}(m_z L_x - m_x L_z) + \frac{1}{2}\lambda(z, z^*)m_y \\
&\quad + \frac{1}{8i}\left(e^{i\phi}\tan\frac{\theta}{2} - e^{i\phi}\cot\frac{\theta}{2}\right)\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right) \\
\frac{1}{2}\left(z_+ \frac{\partial}{\partial z_+} - z_- \frac{\partial}{\partial z_-}\right) &= -\frac{1}{2}L_z - \frac{i}{2}(m_x L_y - m_y L_x) + \frac{1}{2}\lambda(z, z^*)m_z
\end{aligned} \tag{43}$$

where

$$\lambda(z, z^*) = \frac{1}{2}\left(z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}\right)$$

$$m_x = \sin\theta\cos\phi, \qquad m_y = \sin\theta\sin\phi, \qquad m_z = \cos\theta$$

Because we only treat those operators that commute with the total number operator n, the last term proportional to $[z(\partial/\partial z) - z^*(\partial/\partial z^*)]$ vanishes when operated on $\bar{F}^{(N)}$ on $\bar{F}^{(A)}$ due to (41). The orbital angular momentum operators L_x, L_y, and L_z are expressed in terms of the spin in the phase space, $\mathbf{m}$, as

$$L_x = im_z\frac{\partial}{\partial m_y}, \qquad L_y = -im_z\frac{\partial}{\partial m_x}, \qquad L_z = i\left(\frac{\partial}{\partial m_x}m_y - \frac{\partial}{\partial m_y}m_x\right)$$

where m_x and m_y are treated as independent variables. Due to a change of the phase volume, we have to consider the function defined by

$$f^{(\Omega)}(m_x, m_y, t) = (1/m_z)F_J^{(\Omega)}(m_x, m_y, t) \tag{44}$$

then operators L_μ are transformed into

$$l_x = i\frac{\partial}{\partial m_y}m_z, \qquad l_y = -i\frac{\partial}{\partial m_x}, m_z \qquad l_z = i\left(\frac{\partial}{\partial m_x}m_y - \frac{\partial}{\partial m_y}m_x\right) \tag{45}$$

when acting on $f^{(\Omega)}(m_x, m_y, t)$.

If we concern ourselves with the subspace having a fixed magnitude of spin J, the function $\bar{F}^{(N)}(\mathbf{z}, \mathbf{z}^*)$ turns out to be an eigenfunction of $\lambda(z, z^*)$ with an eigenvalue $2J$. Meanwhile, in the case of $\bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*)$, it can be seen that an operation of $\lambda(z, z^*)$ on $\bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*)$ is equivalent to the multiplication of $F_J^{(A)}(\theta, \phi)$ by $-2(J+1)$. This is recognized as follows. By using (34) and integrating by parts, whenever $\lambda(z, z^*)$ appears, it can be replaced by a number $-2(J+1)$:

$$\int \frac{d^2z}{\pi} |z|^{4J+2}\lambda(z, z^*)\bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*)$$
$$= \frac{1}{2}\int \frac{d^2z}{\pi} z^{2J+1}(z^*)^{2J+1}\left(z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}\right)\bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*)$$
$$= -2(J+1)\int \frac{d^2z}{\pi} |z|^{4J+2}\bar{F}^{(A)}(\mathbf{z}, \mathbf{z}^*)$$

It will be convenient to define the following phase space operators:

$$\mathscr{S}^{(N)} = J\mathbf{m} + \tfrac{1}{2}\mathbf{L} - \tfrac{1}{2}i(\mathbf{m} \times \mathbf{L}) \tag{46a}$$

and

$$\mathscr{S}^{(A)} = (J+1)\mathbf{m} + \tfrac{1}{2}\mathbf{L} + \tfrac{1}{2}i(\mathbf{m} \times \mathbf{L}) \tag{46b}$$

Now the product theorem is read in the following way: The c-number equivalents are given by $\mathscr{S}^{(\Omega)}F_J^{(\Omega)}(\theta, \phi)$ for the product operator SG, and $\mathscr{S}^{(\Omega)*}F_J^{(\Omega)}(\theta, \phi)$ for the product operator GS. By induction we are finally led to the following corollary.

Corollary 2. The c-number equivalents $F_{12J}^{(N)}(\theta, \phi)$ and $F_{12J}^{(A)}(\theta, \phi)$ of the product operator G_1G_2 are given by

$$F_{12}^{(N)}(\theta, \phi) = G_1(\mathscr{S}^{(N)})F_2^{(N)}(\theta, \phi) = G_2(\mathscr{S}^{(N)*})F_1^{(N)}(\theta, \phi) \tag{47a}$$

and

$$F_{12}^{(A)}(\theta, \phi) = G_1(\mathscr{S}^{(A)})F_2^{(A)}(\theta, \phi) = G_2(\mathscr{S}^{(A)*})F_1^{(A)}(\theta, \phi) \tag{47b}$$

where $F_1^{(\Omega)}(\theta, \phi)$ and $F_2^{(\Omega)}(\theta, \phi)$ are c-number equivalents of G_1 and G_2, respectively, $G_1(\mathscr{S}^{(\Omega)})$ and $G_2(\mathscr{S}^{(\Omega)})$ being operators in which spin operators are replaced by the phase space operator $\mathscr{S}^{(\Omega)}$ defined by (46a) and (46b).

5. LIOUVILLE EQUATION AND ITS CLASSICAL LIMIT

In order to illustrate the usefulness of the product theorem, we shall consider an equation of motion of a dynamical variable or a density matrix. Let the Hamiltonian of the system be $\mathscr{H}$; then equations of motion for an operator G and a density matrix ρ are written in the form

$$\dot{G} = -i[G, \mathscr{H}] \tag{48a}$$

and

$$\dot{\rho} = -i[\mathscr{H}, \rho] \tag{48b}$$

These equations take the following phase space form:

$$\dot{F}^{(\Omega)}(\theta, \phi) = -i\mathscr{L}^{(\Omega)}F^{(\Omega)}(\theta, \phi) \tag{49a}$$

and

$$\dot{P}^{(\Omega)}(\theta, \phi) = i\mathscr{L}^{(\Omega)}P^{(\Omega)}(\theta, \phi) \tag{49b}$$

where $F^{(\Omega)}(\theta, \phi)$ and $P^{(\Omega)}(\theta, \phi)$ are c-number functions mapped from the operators G and ρ in the Ω-mapping rule, respectively. As a simple application of the results of the preceding section, the Liouville operator in (49a) and (49b) is given by

$$\mathscr{L}^{(\Omega)} = \mathscr{H}(\mathscr{S}^{(\Omega)*}) - \mathscr{H}(\mathscr{S}^{(\Omega)}) \tag{50}$$

The classical limit of $\mathscr{L}^{(\Omega)}$ is obtained by replacing $\mathscr{S}^{(\Omega)}$ with $J\mathbf{m} + \frac{1}{2}\mathbf{L}$ and retaining only linear terms in $\mathbf{L}$.

A few examples may serve for illustration.

5.1. A Spin Under the Influence of a Static Magnetic Field Along the z Axis

The Hamiltonian is given by

$$\mathscr{H} = -\omega_0 S_z \tag{51}$$

and the corresponding Liouville operator is written in the form

$$\mathscr{L}^{(\Omega)} = -\omega_0(S_z^{(\Omega)*} - S_z^{(\Omega)}) = \omega_0 L_z = -i\omega_0\, \partial/\partial\phi \tag{52}$$

5.2. Heisenberg Model

The Hamiltonian of the system can be written in the form

$$\mathscr{H} = -\sum_{\langle i,j\rangle} \mathscr{J}_{ij}\mathbf{S}_i\cdot\mathbf{S}_j \tag{53}$$

Using (50), we find the Liouville equation for the anti-normal association rule in the form

$$\dot{P}^{(\mathrm{A})}(\theta, \phi) = i\mathscr{L}^{(\mathrm{A})}P^{(\mathrm{A})}(\theta, \phi) \tag{54}$$

where

$$\begin{aligned}\mathscr{L}^{(\mathrm{A})} &= -\sum_{\langle i,j\rangle} \mathscr{J}_{ij}\{\mathscr{S}_i^{(\mathrm{A})*}\cdot\mathscr{S}_j^{(\mathrm{A})*} - \mathscr{S}_i^{(\mathrm{A})}\cdot\mathscr{S}_j^{(\mathrm{A})}\} \\ &= 2\sum_{\langle i,j\rangle} \mathscr{J}_{ij}\mathbf{L}_i\cdot\{(J+1)\mathbf{m}_j + \tfrac{1}{2}i\mathbf{m}_j \times \mathbf{L}_j\}\end{aligned} \tag{55}$$

Transformation of the variables θ and ϕ into m_x and m_y yields an equation of the following form:

$$\dot{f}^{(\mathrm{A})}(m_x, m_y, t) = iL^{(\mathrm{A})}f^{(\mathrm{A})}(m_x, m_y, t) \tag{56}$$

In particular, for the classical limit, we have

$$iL_{\mathrm{cl}}^{(\mathrm{A})} = J \sum \mathscr{J}_{ij} \left\{ \frac{\partial}{\partial m_x^i} (m_y^i m_z^j - m_z^i m_z^j) + \frac{\partial}{\partial m_y^i} (m_z^i m_x^j - m_x^i m_z^j) \right\}$$

where the function $f^{(\mathrm{A})}(m_x, m_y, t)$ has been defined by (44). A characteristic curve gives immediately a classical trajectory of the form

$$\dot{\mathbf{M}}_i = -\sum \mathscr{J}_{ij} \mathbf{M}_i \times \mathbf{M}_j$$

where $\mathbf{M}_i = J\mathbf{m}_i$.

6. SUMMARY AND CONCLUSION

We have established a generalized phase space method for spin systems. We summarize the results as follows.

1. Mapping rules from spin operators onto a c-number space are obtained by generalizing those for boson operators using Schwinger's coupled boson representation.
2. We obtain the normal and the anti-normal representations of the spin operators unambiguously, by considering the transformation properties of the spin coherent state.
3. The product theorem is established. Making use of the theorem, we can obtain the c-number description of the Liouville equation; the classical limit of the spin operators is also examined.

Thus we have found the c-number association rules in a quite general manner.

Our method may have wide applicability: It is not confined to single-spin problems, but can be applied to many-body problems. Indeed, we have obtained the basic equation for the Heisenberg magnet in Section 5; using this equation, we can discuss spin waves, critical dynamics, and so on. These are left for future study.

In forthcoming papers, the relation of the various distribution functions and the fluctuation phenomena in a superradiant system is investigated.

APPENDIX A. PROOF OF THEOREM 1

Let $F_1^{(\mathrm{N})}(\mathbf{z}, \mathbf{z}^*)$ and $F_2^{(\mathrm{A})}(\mathbf{z}, \mathbf{z}^*)$ be c-number functions corresponding to operators G_1 and G_2, respectively. Then the trace of the product G_1G_2 can be written in the form

$$\mathrm{Tr}\, G_1G_2 = \int \frac{d^2z}{\pi^2} F_1^{(\mathrm{N})}(\mathbf{z}, \mathbf{z}^*) F_2^{(\mathrm{A})}(\mathbf{z}, \mathbf{z}^*)$$

If we use (32), we can extract the component of the irreducible representation of rank J:

$$(\mathrm{Tr}\, G_1 G_2)_J = \int \frac{d^2\mathbf{z}}{\pi^2} (\exp - |z|^2) \frac{|z|^{4J}}{(2J)!} F_{1,J}^{(N)}(\theta, \phi) F_{2,J}^{(A)}(\theta, \phi)$$

After integrating over z, we obtain Theorem 1.

APPENDIX B. SOME EXAMPLES OF c-NUMBER FUNCTIONS

In this appendix we show how the spin operators are mapped onto a c-number space in the framework of our theory. The following examples may serve for illustration.

(i) $G = S_z$. Since the spin operator S_z is expressed in terms of Bose operators as

$$S_z = \tfrac{1}{2}(a_+{}^\dagger a_+ - a_-{}^\dagger a_-) = \tfrac{1}{2}(a_+ a_+{}^\dagger - a_- a_-{}^\dagger)$$

its c-number equivalent is given by

$$F^{(N)}(\mathbf{z}, \mathbf{z}^*) = F^{(A)}(\mathbf{z}, \mathbf{z}^*) = \tfrac{1}{2}(|z_+|^2 - |z_-|^2) = \tfrac{1}{2}|z|^2 \cos\theta$$

Extracting the factor $(\exp - |z|^2)|z|^{4J}/(2J)!$, we obtain $F_J^{(N)}(\theta, \phi)$ as follows:

$$F^{(N)}(\mathbf{z}, \mathbf{z}^*) = (\exp - |z|^2) \sum_J \frac{|z|^{4J-2}}{(2J-1)!} \tfrac{1}{2}|z|^2 \cos\theta$$
$$= (\exp - |z|^2) \sum_J \frac{|z|^{4J}}{(2J)!} J \cos\theta$$

Hence

$$F_J^{(N)}(\theta, \phi) = J \cos\theta$$

Using (34), $F_J^{(A)}(\theta, \phi)$ is also given by

$$F_J^{(A)}(\theta, \phi) = \int \frac{(\exp - |z|^2)}{(2J)!} |z|^{4J+4} \frac{1}{2} \cos\theta \frac{d^2 z}{\pi} = (J + 1)\cos\theta$$

(ii) $G = e^{\gamma S_z}$. Using the transformation properties of the spin coherent states, $F^{(N)}(\mathbf{z}, \mathbf{z}^*)$ and $F^{(A)}(\mathbf{z}, \mathbf{z}^*)$ are given by

$$F^{(N)}(\mathbf{z}, \mathbf{z}^*) = \exp\{\cosh \tfrac{1}{2}\gamma + \cos\theta \sinh \tfrac{1}{2}\gamma)|z|^2 - |z|^2\}$$

and

$$F^{(A)}(\mathbf{z}, \mathbf{z}^*) = \exp\{|z|^2 - (\cosh \tfrac{1}{2}\gamma - \cos\theta \sinh \tfrac{1}{2}\gamma)|z|^2\}$$

respectively; the corresponding functions in the subspace of J, $F_J^{(N)}(\theta, \phi)$

Table I. The c-Number Equivalents for Some Operators

G	$F^{(N)}(\mathbf{z}, \mathbf{z}^*)$	$F_J^{(N)}(\theta, \phi)$
$e^{\xi S_x}$	$\exp[\cosh \frac{1}{2}\xi \vert z\vert^2 + \sinh \frac{1}{2}\xi (z_+^* z_- + z_-^* z_+)] \exp(-\vert z\vert^2)$	$(\cosh \frac{1}{2}\xi + \sinh \frac{1}{2}\xi \sin\theta \cos\phi)^{2J}$
$e^{\xi S_y}$	$\exp[\cosh \frac{1}{2}\xi \vert z\vert^2 - i \sinh \frac{1}{2}\xi (z_+^* z_- - z_-^* z_+)] \exp(-\vert z\vert^2)$	$(\cosh \frac{1}{2}\xi + \sinh \frac{1}{2}\xi \sin\theta \sin\phi)^{2J}$
$e^{\xi S_z}$	$\exp(e^{\xi/2}\vert z_+\vert^2 + e^{-\xi/2}\vert z_-\vert^2) \exp(-\vert z\vert^2)$	$(\cosh \frac{1}{2}\xi + \sinh \frac{1}{2}\xi \cos\theta)^{2J}$
G	$F^{(A)}(\mathbf{z}, \mathbf{z}^*)$	$F_J^{(A)}(\theta, \phi)$
$e^{\xi S_x}$	$\exp[-\cosh \frac{1}{2}\xi \vert z\vert^2 + \sinh \frac{1}{2}\xi (z_+^* z_- + z_-^* z_+)] \exp(\vert z\vert^2)$	$(\cosh \frac{1}{2}\xi - \sinh \frac{1}{2}\xi \sin\theta \cos\phi)^{-2(J+1)}$
$e^{\xi S_y}$	$\exp[-\cosh \frac{1}{2}\xi \vert z\vert^2 - i \sinh \frac{1}{2}\xi (z_+^* z_- - z_-^* z_+)] \exp(\vert z\vert^2)$	$(\cosh \frac{1}{2}\xi - \sinh \frac{1}{2}\xi \sin\theta \sin\phi)^{-2(J+1)}$
$e^{\xi S_z}$	$\exp(-e^{-\xi/2}\vert z_+\vert^2 - e^{\xi/2}\vert z_-\vert) \exp(\vert z\vert^2)$	$(\cosh \frac{1}{2}\xi - \sinh \frac{1}{2}\xi \cos\theta)^{-2(J+1)}$

and $F_J^{(A)}(\theta, \phi)$, are obtained from (32) and (34) as

$$F_J^{(N)}(\theta, \phi) = (\cosh \tfrac{1}{2}\gamma + \cos\theta \sinh \tfrac{1}{2}\gamma)^{2J}$$

and

$$F_J^{(A)}(\theta, \phi) = (\cosh \tfrac{1}{2}\gamma - \cos\theta \sinh \tfrac{1}{2}\gamma)^{-2(J+1)}$$

Other examples are shown in Table 1.

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